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A perturbation method for dark solitons based on a complete set of the squared Jost solutions

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Abstract
A perturbation method for dark solitons is developed, which is based on the construction and the rigorous proof of the complete set of squared Jost solutions. The general procedure solving the adiabatic solution of perturbed nonlinear Schrödinger[+] equation, the time-evolution equation of dark soliton parameters and a formula for calculating the first-order correction are given. The method can also overcome the difficulties resulting from the non-vanishing boundary condition.

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1. Introduction

In recent decades, the temporal and the spatial optical solitons are important subjects in theory as well in experiment. Under ideal conditions, the evolutions of the temporal optical solitons in fibres [1] and the spatial optical solitons in waveguides [2] are governed by the well-known nonlinear Schrödinger (NLS) equation

$$\mathrm{i}p_t - \sigma p_{xx} + 2|p|^2 p = 0, \qquad \sigma = \pm 1. \tag{1}$$

In the case of $\sigma = -1$, i.e., the case of the abnormal group-velocity dispersion for temporal solitons or the self-focusing media for spatial solitons, equation (1) has bright soliton solutions under vanishing boundary conditions; in the case of $\sigma = 1$, i.e., the case of the normal group-velocity dispersion for temporal solitons or the self-defocusing media for spatial solitons, equation (1) has dark soliton solutions under non-vanishing boundary conditions [3]. However, there are always high-order effects in more realistic situations, which are usually treated as small perturbations.

For bright soliton ($\sigma = -1$) perturbation under vanishing boundary conditions, two systematic perturbation methods, the method based upon the inverse scattering transform (IST) [4–6] and the direct method based upon the theory of linear partial differential equations [7–10] have been well established and developed for completely and nearly integrable systems, respectively.

However, in the presence of perturbations, it is very difficult to directly generalize the above perturbation theories to the dark soliton case ($\sigma = 1$) because of the non-vanishing boundary conditions resulting in problems such as the time dependence of the boundary values, the background energy divergence and so on [11]. Thus, no one has so far attempted to develop a perturbation method for dark solitons based on IST. Several analytical works have treated the perturbation-induced dynamics of dark solitons [12, 13], but the approaches used there are not systematic. The direct perturbation methods for dark solitons were tried by Konotop *et al* [14] and by Chen *et al* [11], respectively, but these were not successful in general. Konotop *et al* did not deal at all with the evolution of the background (non-vanishing boundary), the most crucial and difficult point of the perturbation theory for dark solitons, and some results were also incorrect [11]. Although the method in [11] was practicable, it was clear that the authors incorrectly discarded a continuous spectrum basic vector (the complete sets should have two not just one continuous spectrum basic vector), which resulted in the proof for the completeness of the squared Jost solutions actually being incorrect and not strict. Besides, there were some errors within the orthogonality relationships and the discrete spectra in [11]. In addition to the above attempts, the adiabatic method [15, 16], which is based upon an attempt to separate the background from the perturbed NLS$^+$ equation, has also been proposed. Since the phase difference between each end of the boundary values exists (see section 2), such a separation is impossible. Also, the method has a difficulty in self-consistency [17]. Hence, the perturbation for dark solitons is still an open problem, that is, the systematic perturbation theory for dark solitons has never been successfully established up to now.

In this paper, we introduce the squared Jost solutions and define their adjoint states and the corresponding inner products in a manner similar to that in [11]. Using the explicit expression of squared Jost solutions and the residue theorem, the completeness and the orthogonality of the squared Jost solution sets will be proved directly and strictly. By expanding and solving the first-order linearized inhomogeneous equation, the adiabatic solution of the perturbed NLS$^+$ equation, the evolution equations of soliton parameters and the formula for calculating the first-order correction will be obtained. The problem of the damping NLS$^+$ equation will be studied as an important example.

2. NLS$^+$ equation and some results of the IST method

For the case of dark solitons ($\sigma = 1$), let $p = u\,\mathrm{e}^{\mathrm{i}2\rho^2 t}$, in which ρ is a positive constant, equation (1) becomes

$$\mathrm{i}u_t - u_{xx} + 2(|u|^2 - \rho^2)u = 0, \tag{2}$$

which is the well-known NLS$^+$ equation. Under the non-vanishing boundary conditions, i.e.,

$$u \longrightarrow \begin{cases} \rho, & x \longrightarrow +\infty, \\ \rho\,\mathrm{e}^{\mathrm{i}\theta}, & x \longrightarrow -\infty, \end{cases} \tag{3}$$

equation (2) has dark soliton solutions, e.g., the single-soliton solution

$$u(x,t) = \mathrm{e}^{-\mathrm{i}\beta_1}\{\lambda_1 + \mathrm{i}k_1 \tanh\theta_1\}, \tag{4}$$

where θ is the phase difference between each end of the boundary values, β_1, λ_1 and k_1 are the parameters of the dark soliton, and

$$\zeta_1 = \lambda_1 + \mathrm{i}k_1 = \rho\,\mathrm{e}^{\mathrm{i}\beta_1}, \tag{5}$$

$$\theta_1 = k_1(x - x_1 - 2\lambda_1 t). \tag{6}$$

Under non-vanishing boundary conditions, the Jost solutions of equation (2) are well known [18, 19], and we just list some results which are helpful in constructing the direct perturbation theory for dark solitons. The corresponding Lax equations of equation (2) are

$$\partial_x \Phi(\lambda) = L(\lambda)\Phi(\lambda), \tag{7}$$

$$\partial_t \Phi(\lambda) = M(\lambda)\Phi(\lambda), \tag{8}$$

in which the Lax pair is

$$L(\lambda) = -\mathrm{i}\lambda\sigma_3 + U, \tag{9}$$

$$M(\lambda) = \mathrm{i}2\lambda^2\sigma_3 - 2\lambda U + \mathrm{i}(U^2 - \rho^2 + U_x)\sigma_3, \tag{10}$$

σ_j $(j = 1, 2, 3)$ are Pauli matrices, and

$$U = \begin{pmatrix} 0 & u \\ \overline{u} & 0 \end{pmatrix} \tag{11}$$

with $\overline{u}$ being the complex conjugate of u (in what follows, the bar denotes complex conjugate). The auxiliary parameter ζ can be introduced to make λ and $\kappa = \sqrt{\lambda^2 - \rho^2}$ become single-valued functions of ζ, i.e.,

$$\lambda = \tfrac{1}{2}(\zeta + \rho^2\zeta^{-1}) \qquad \text{and} \qquad \kappa = \tfrac{1}{2}(\zeta - \rho^2\zeta^{-1}). \tag{12}$$

With asymptotic solutions of equation (7), $E(x, \zeta)$ $(x \longrightarrow +\infty)$ and $E_-(x, \zeta)$ $(x \longrightarrow -\infty)$, the usual Jost solutions are defined as

$$\Psi(x, \zeta) = (\widetilde{\psi}(x, \zeta), \psi(x, \zeta)) \longrightarrow E(x, \zeta), \qquad x \longrightarrow +\infty, \tag{13}$$

$$\Phi(x, \zeta) = (\phi(x, \zeta), \widetilde{\phi}(x, \zeta)) \longrightarrow E_-(x, \zeta), \qquad x \longrightarrow -\infty, \tag{14}$$

where

$$E_-(x, \zeta) = \mathrm{e}^{\frac{1}{2}\mathrm{i}\theta\sigma_3} E(x, \zeta), \tag{15}$$

$$E(x, \zeta) = \begin{pmatrix} 1 & -\mathrm{i}\rho\zeta^{-1} \\ \mathrm{i}\rho\zeta^{-1} & 1 \end{pmatrix} \mathrm{e}^{-\mathrm{i}\kappa x \sigma_3}. \tag{16}$$

$\Phi(x, \zeta)$ and $\Psi(x, \zeta)$ are not linearly independent:

$$\phi(x, \zeta) = a(\zeta)\widetilde{\psi}(x, \zeta) + b(\zeta)\psi(x, \zeta), \tag{17}$$

$$\widetilde{\phi}(x, \zeta) = \widetilde{b}(\zeta)\widetilde{\psi}(x, \zeta) + \widetilde{a}(\zeta)\psi(x, \zeta). \tag{18}$$

$\psi(x, \zeta)$, $\phi(x, \zeta)$ and $a(\zeta)$ can be analytically continued to the upper half-plane of complex ζ, while $\widetilde{\psi}(x, \zeta)$, $\widetilde{\phi}(x, \zeta)$ and $\widetilde{a}(\zeta)$ can be analytically continued to the lower half-plane of complex ζ. Usually $b(\zeta)$ and $\widetilde{b}(\zeta)$ cannot be analytically continued outside the real axes.

Since two values of ζ correspond to a single value of λ, under the transformation $\zeta \longrightarrow \rho^2\zeta^{-1}$, the Jost solutions have the following so-called reduction relations:

$$\widetilde{\psi}(x, \rho^2\zeta^{-1}) = \mathrm{i}\rho^{-1}\zeta\psi(x, \zeta), \qquad \psi(x, \rho^2\zeta^{-1}) = -\mathrm{i}\rho^{-1}\zeta\widetilde{\psi}(x, \zeta), \tag{19}$$

$$\phi(x, \rho^2\zeta^{-1}) = \mathrm{i}\rho^{-1}\zeta\widetilde{\phi}(x, \zeta), \qquad \widetilde{\phi}(x, \rho^2\zeta^{-1}) = -\mathrm{i}\rho^{-1}\zeta\phi(x, \zeta), \tag{20}$$

$$\widetilde{a}(\rho^2\zeta^{-1}) = a(\zeta), \qquad \widetilde{b}(\rho^2\zeta^{-1}) = -b(\zeta) \quad (\zeta \text{ is real}). \tag{21}$$

The zeros ζ_n of $a(\zeta)$ are located on the half-circle of radius ρ centred at the origin, that is

$$\zeta_n = \rho\,\mathrm{e}^{\mathrm{i}\beta_n}, \qquad 0 < \beta_n < \pi \tag{22}$$

or

$$\zeta_n = \lambda_n + \mathrm{i}k_n, \qquad k_n > 0, \quad n = 1, 2, \ldots. \tag{23}$$

In the case of non-reflection,

$$\phi(x,\zeta) = a(\zeta)\widetilde{\psi}(x,\zeta), \qquad \widetilde{\phi}(x,\zeta) = \widetilde{a}(\zeta)\psi(x,\zeta), \tag{24}$$

$$a(\zeta) = \mathrm{e}^{\frac{\mathrm{i}}{2}\theta}\prod_{n=1}^{N}\frac{\zeta-\zeta_n}{\zeta-\overline{\zeta}_n}, \qquad \theta = -2\sum_{n=1}^{N}\beta_n. \tag{25}$$

When $a(\zeta)$ vanishes only at $\zeta = \zeta_1$, the usual IST method yields the single-soliton solution (4) and the corresponding Jost solutions

$$\phi(x,t,\zeta) = \begin{pmatrix}\phi_1(x,t,\zeta)\\ \phi_2(x,t,\zeta)\end{pmatrix} = \begin{pmatrix}\zeta-\zeta_1+\mathrm{i}k_1\operatorname{sech}\theta_1\,\mathrm{e}^{-\theta_1}\\ \mathrm{i}\rho\zeta^{-1}(\zeta-\zeta_1)-\mathrm{e}^{\mathrm{i}\beta_1}k_1\operatorname{sech}\theta_1\,\mathrm{e}^{-\theta_1}\end{pmatrix}\frac{\mathrm{e}^{-\mathrm{i}\beta_1}\,\mathrm{e}^{-\mathrm{i}\kappa x}}{\zeta-\overline{\zeta}_1}, \tag{26}$$

$$\psi(x,t,\zeta) = \begin{pmatrix}\psi_1(x,t,\zeta)\\ \psi_2(x,t,\zeta)\end{pmatrix} = \begin{pmatrix}-\mathrm{i}\rho\zeta^{-1}(\zeta-\overline{\zeta}_1)-\mathrm{e}^{-\mathrm{i}\beta_1}k_1\operatorname{sech}\theta_1\,\mathrm{e}^{-\theta_1}\\ \zeta-\overline{\zeta}_1-\mathrm{i}k_1\operatorname{sech}\theta_1\,\mathrm{e}^{-\theta_1}\end{pmatrix}\frac{\mathrm{e}^{\mathrm{i}\kappa x}}{\zeta-\overline{\zeta}_1}. \tag{27}$$

In order to satisfy the second Lax equation (8), the Jost solutions should be corrected as

$$\phi(x,t,\zeta) \longrightarrow h(t,\zeta)\phi(x,t,\zeta), \qquad \widetilde{\phi}(x,t,\zeta) \longrightarrow h^{-1}(t,\zeta)\widetilde{\phi}(x,t,\zeta), \tag{28}$$

$$\widetilde{\psi}(x,t,\zeta) \longrightarrow h(t,\zeta)\widetilde{\psi}(x,t,\zeta), \qquad \psi(x,t,\zeta) \longrightarrow h^{-1}(t,\zeta)\psi(x,t,\zeta), \tag{29}$$

with $h(t,\zeta) = \mathrm{e}^{\mathrm{i}2\kappa\lambda t}$.

3. The perturbed NLS$^+$ equation and squared Jost solutions

3.1. The linearization of the perturbed NLS$^+$ equation

The perturbed NLS$^+$ equation can be written as

$$\mathrm{i}v_t - v_{xx} + 2(|v|^2-\rho^2)v = \epsilon r[v], \tag{30}$$

where ϵ is a small parameter measuring the weakness of the perturbation ($0 < \epsilon \ll 1$) and $r[v]$ is a known functional of v for most higher order effects of dark solitons. Consider an approximate solution of equation (30) to first order:

$$v = u + \epsilon q \tag{31}$$

with the initial condition

$$v(x,0) = u(x,0), \qquad q(x,0) = 0, \tag{32}$$

where $u(x,0)$ is the exact soliton solution of the unperturbed NLS$^+$ equation at $t = 0$. According to the idea of adiabatic solution, the zeroth-order approximation $u = u(x,t)$ cannot be the exact solution of equation (2), that is, the free parameters in u must evolve with the temporal scale of ϵt. Introducing the multiscale expansion of time

$$\partial_t = \sum_{n=0}^{\infty}\epsilon^n\partial_{t_n} \tag{33}$$

with $t_n = \epsilon^n t$ $(n = 0, 1, 2, \ldots)$ being treated as independent as usual, substituting equations (31), (33) into equation (30) and using equation (32), we get

$$\mathrm{i}u_{t_0} - u_{xx} + 2(|u|^2-\rho^2)u = 0 \tag{34}$$

and the linearized equation (to first order of ϵ)

$$\mathrm{i}q_{t_0} - q_{xx} + 2(2|u|^2 - \rho^2)q + 2u^2\overline{q} = R[u], \qquad q(x,0) = 0, \tag{35}$$

in which $R \equiv R[u] = r[u] - \mathrm{i}u'$ is the effective source, the prime means a derivative with respect to t_1. Combining equation (35) with its complex conjugate equation, we have

$$\left\{\mathrm{i}\partial_{t_0} - \mathbf{L}(u)\right\}\mathbf{q} = \mathbf{R}, \tag{36}$$

in which

$$\mathbf{L}(u) = \begin{pmatrix} \partial_x^2 - 2(2|u|^2 - \rho^2) & -2u^2 \\ 2\overline{u}^2 & -\partial_x^2 + 2(2|u|^2 - \rho^2) \end{pmatrix}, \tag{37}$$

$$\mathbf{q} = \begin{pmatrix} q \\ \overline{q} \end{pmatrix}, \qquad \mathbf{R} = \begin{pmatrix} R \\ -\overline{R} \end{pmatrix}. \tag{38}$$

It is known that in order to solve equation (36), one must first solve its corresponding homogeneous equation

$$\left\{\mathrm{i}\partial_{t_0} - \mathbf{L}(u)\right\}\mathbf{q} = 0. \tag{39}$$

For the single-soliton case, it is convenient to discuss the problem in a tracing frame of reference (TFR) which is moving with the soliton. Then making the transformation

$$t_0, \qquad x \longrightarrow t_0, \qquad z = x - x_1 - 2\lambda_1 t_0, \tag{40}$$

$$\partial_{t_0}, \qquad \partial_x \longrightarrow \partial_{t_0} - 2\lambda_1\partial_z, \qquad \partial_z, \tag{41}$$

equation (39) becomes

$$\left\{\mathrm{i}\partial_{t_0} - \mathbf{L}(z)\right\}\mathbf{q} = 0, \tag{42}$$

where

$$\mathbf{L}(z) = \begin{pmatrix} \partial_z^2 + \mathrm{i}2\lambda_1\partial_z - 2(2|u|^2 - \rho^2) & -2u^2 \\ 2\overline{u}^2 & -\partial_z^2 + \mathrm{i}2\lambda_1\partial_z + 2(2|u|^2 - \rho^2) \end{pmatrix}. \tag{43}$$

3.2. The squared Jost solutions

Now let us solve the eigenvalue problem of operator $\mathbf{L}(z)$. For this purpose, we define the squared Jost solutions as

$$\Phi(z,\zeta) = \begin{pmatrix} \phi_1^2(z,\zeta) \\ \phi_2^2(z,\zeta) \end{pmatrix} = \exp(\mathrm{i}2\kappa(x_1 + 2\lambda_1 t_0)) \begin{pmatrix} \phi_1^2(x,t_0,\zeta) \\ \phi_2^2(x,t_0,\zeta) \end{pmatrix}, \tag{44}$$

$$\widetilde{\Phi}(z,\zeta) = \begin{pmatrix} \widetilde{\phi}_1^2(z,\zeta) \\ \widetilde{\phi}_2^2(z,\zeta) \end{pmatrix} = \exp(-\mathrm{i}2\kappa(x_1 + 2\lambda_1 t_0)) \begin{pmatrix} \widetilde{\phi}_1^2(x,t_0,\zeta) \\ \widetilde{\phi}_2^2(x,t_0,\zeta) \end{pmatrix}, \tag{45}$$

$$\Psi(z,\zeta) = \begin{pmatrix} \psi_1^2(z,\zeta) \\ \psi_2^2(z,\zeta) \end{pmatrix} = \exp(-\mathrm{i}2\kappa(x_1 + 2\lambda_1 t_0)) \begin{pmatrix} \psi_1^2(x,t_0,\zeta) \\ \psi_2^2(x,t_0,\zeta) \end{pmatrix}, \tag{46}$$

$$\widetilde{\Psi}(z,\zeta) = \begin{pmatrix} \widetilde{\psi}_1^2(z,\zeta) \\ \widetilde{\psi}_2^2(z,\zeta) \end{pmatrix} = \exp(\mathrm{i}2\kappa(x_1 + 2\lambda_1 t_0)) \begin{pmatrix} \widetilde{\psi}_1^2(x,t_0,\zeta) \\ \widetilde{\psi}_2^2(x,t_0,\zeta) \end{pmatrix} \tag{47}$$

in a manner similar to that in [11]. Letting $H(t_0,\zeta) = h^2(t_0,\zeta)\exp(-i2\kappa(x_1 + 2\lambda_1 t_0))$, it is not difficult to show that $H(t_0,\zeta)\Phi(z,\zeta)$, $H^{-1}(t_0,\zeta)\widetilde{\Phi}(z,\zeta)$, $H^{-1}(t_0,\zeta)\Psi(z,\zeta)$ and $H(t_0,\zeta)\widetilde{\Psi}(z,\zeta)$ are all solutions of the homogeneous equation (42), that is, substituting them into equation (42) respectively, the following eigenvalue equations can be obtained

$$\mathbf{L}(z)\Phi(z,\zeta) = -4\kappa(\lambda - \lambda_1)\Phi(z,\zeta), \tag{48}$$
$$\mathbf{L}(z)\widetilde{\Phi}(z,\zeta) = 4\kappa(\lambda - \lambda_1)\widetilde{\Phi}(z,\zeta), \tag{49}$$
$$\mathbf{L}(z)\Psi(z,\zeta) = 4\kappa(\lambda - \lambda_1)\Psi(z,\zeta), \tag{50}$$
$$\mathbf{L}(z)\widetilde{\Psi}(z,\zeta) = -4\kappa(\lambda - \lambda_1)\widetilde{\Psi}(z,\zeta), \tag{51}$$

which shows that the squared Jost solutions (44)–(47) are exactly the eigenfunctions of the linear operator $\mathbf{L}(z)$.

3.3. The complete set of linear operator $\mathbf{L}(z)$

As in the case of bright solitons, the squared Jost solutions (44)–(47) can be used to construct the basic vectors of a complete set. However, we know that $\widetilde{\Phi}(z,\zeta)$, $\widetilde{\Psi}(z,\zeta)$ and $\Phi(z,\zeta)$, $\Psi(z,\zeta)$ are not linear independent from the following reduction relations of squared Jost solutions

$$\widetilde{\Phi}(z,\rho^2\zeta^{-1}) = -\rho^{-2}\zeta^2\Phi(z,\zeta), \tag{52}$$
$$\widetilde{\Psi}(z,\rho^2\zeta^{-1}) = -\rho^{-2}\zeta^2\Psi(z,\zeta), \tag{53}$$

which can be derived from equations (19)–(21) and equation (24). Hence, it is practicable to only take the continuous spectra $\Phi(z,\zeta)$ and $\Psi(z,\zeta)$ as the continuously spectrum basic vectors, that is, there are only two independent eigenvalue equations (48) and (50) for the operator $\mathbf{L}(z)$. The explicit expressions of $\Phi(z,\zeta)$ and $\Psi(z,\zeta)$ can be found in appendix A.

To solve the adjoint eigenvalue problem of the operator $\mathbf{L}(z)$, we first define directly the adjoint states of $\Phi(z,\zeta)$ and $\Psi(z,\zeta)$ as

$$\Phi(z,\zeta)^A = \Psi(z,\zeta)^T(i\sigma_2), \tag{54}$$
$$\Psi(z,\zeta)^A = \Phi(z,\zeta)^T(i\sigma_2), \tag{55}$$

where $\Phi(z,\zeta)^T$ denotes the transposed matrix of $\Phi(z,\zeta)$. From equations (48), (50) and (54)–(55), it is easy to obtain the following eigenvalue equations

$$\Phi(z,\zeta)^A\mathbf{L}^A(z) = -4\kappa(\lambda - \lambda_1)\Phi(z,\zeta)^A, \tag{56}$$
$$\Psi(z,\zeta)^A\mathbf{L}^A(z) = 4\kappa(\lambda - \lambda_1)\Psi(z,\zeta)^A, \tag{57}$$

in which the adjoint operator $\mathbf{L}^A(z)$ of $\mathbf{L}(z)$ is defined as $\mathbf{L}^A(z) = -\sigma_2\mathbf{L}^T(z)\sigma_2$. Using the continuous eigenstates of $\mathbf{L}(z)$ and $\mathbf{L}^A(z)$, we define the inner product

$$\langle\Phi(z,\zeta')|\Phi(z,\zeta)\rangle = \int_{-\infty}^{\infty} dz\, \Phi(z,\zeta')^A\Phi(z,\zeta). \tag{58}$$

By means of the proof of the completeness and the orthogonality, we find the discrete spectra of the complete set as

$$\Phi_0(z) = \Phi(z,\zeta_1), \qquad \Phi_1(z) = 2k_1[\zeta_1\dot{\Psi}(z,\zeta_1) + \Psi(z,\zeta_1)], \tag{59}$$
$$\Psi_0(z)^A = -e^{i2\beta_1}\Phi(z,\zeta_1)^A, \qquad \Psi_1(z)^A = -e^{i2\beta_1}2k_1[\zeta_1\dot{\Psi}(z,\zeta_1)^A + \Psi(z,\zeta_1)^A], \tag{60}$$

where $\dot{\Psi}(z,\zeta_1)$, etc are the values of corresponding functions at $\zeta = \zeta_1$, the dots denote derivatives with respect to ζ. The discrete spectra $\Phi_0(z)$ and $\Psi_0(z)^A$ are the eigenstates of $\mathbf{L}(z)$ and $\mathbf{L}^A(z)$ respectively, i.e.,

$$\mathbf{L}(z)\Phi_0(z) = 0, \qquad \Psi_0(z)^A\mathbf{L}^A(z) = 0, \tag{61}$$

while $\Phi_1(z)$ and $\Psi_1(z)^A$ satisfy the following equations:

$$\mathbf{L}(z)\Phi_1(z) = 8k_1^3\Phi_0(z), \qquad \Psi_1(z)^A\mathbf{L}^A(z) = 8k_1^3\Psi_0(z)^A. \tag{62}$$

In appendix B, it is proved that $\{\Phi(z,\zeta), \Psi(z,\zeta), \Phi_0(z), \Phi_1(z)\}$ and $\{\Phi(z,\zeta)^A, \Psi(z,\zeta)^A, \Psi_0(z)^A, \Psi_1(z)^A\}$ construct a complete set and the corresponding completeness relationship is expressed as

$$\frac{1}{2\pi}\int_C \frac{\mathrm{d}\zeta}{a(\zeta)^2(1-\rho^4\zeta^{-4})}\{\Psi(z,\zeta)\Psi(z',\zeta)^A - \Phi(z,\zeta)\Phi(z',\zeta)^A\} + \Phi_0(z)\Psi_1(z')^A + \Phi_1(z)\Psi_0(z')^A = I\delta(z-z'), \tag{63}$$

where C denotes a line on the upper half-plane of ζ, from $-\infty + \mathrm{i}0^+$ to $+\infty + \mathrm{i}0^+$, and I is a unit matric. The following orthogonal relations of the complete set are also proved in appendix C:

$$\langle\Phi(z,\zeta')|\Phi(z,\zeta)\rangle = -a(\zeta)^2 2\pi(1-\rho^4\zeta^{-4})\delta(\zeta'-\zeta), \tag{64}$$

$$\langle\Psi(z,\zeta')|\Psi(z,\zeta)\rangle = a(\zeta)^2 2\pi(1-\rho^4\zeta^{-4})\delta(\zeta'-\zeta), \tag{65}$$

$$\langle\Phi(z,\zeta')|\Psi(z,\zeta)\rangle = \langle\Psi(z,\zeta')|\Phi(z,\zeta)\rangle = 0, \tag{66}$$

$$\langle\Psi_0(z)|\Phi(z,\zeta)\rangle = \langle\Psi_0(z)|\Psi(z,\zeta)\rangle = 0, \tag{67}$$

$$\langle\Psi_1(z)|\Phi(z,\zeta)\rangle = \langle\Psi_1(z)|\Psi(z,\zeta)\rangle = 0, \tag{68}$$

$$\langle\Psi_0(z)|\Phi_0(z)\rangle = \langle\Psi_1(z)|\Phi_1(z)\rangle = 0, \tag{69}$$

$$\langle\Psi_0(z)|\Phi_1(z)\rangle = \langle\Psi_1(z)|\Phi_0(z)\rangle = 1. \tag{70}$$

Comparing our results with that in [11], first, we find that the completeness relationship (86) in [11] did not contain the term with $\Psi(z,\zeta)\Psi(z',\zeta)^A$ in the left-hand side of equation (63) in this paper, which certainly resulted in the incorrect proof of the completeness and was a critical mistake in [11]. The reason resulting in the mistake is mainly that the authors choose only one continuous spectrum $\Phi(z,\zeta)$ to form their set of basic vectors; while in our treatment there are two continuous spectra $\Phi(z,\zeta)$ and $\Psi(z,\zeta)$ for the nature of the problem with two independent eigenvalue equations, each being two dimensional. Secondly, the orthogonal relations (64)–(70) in this paper are different from the relations (75), (78)–(81) in [11]. The main difference between them is that our normalized coefficients include the factor $(1-\rho^4\zeta^{-4})$ in contrast to $(1-\rho^2\zeta^{-2})$ in [11]. Thirdly, the discrete spectra $\{\Phi(z,\zeta_1), \dot{\Phi}(z,\zeta_1)\}$ in [11] are different from equation (59) in this paper, which can lead to wrong evolution equations of soliton parameters in [11]. Using the so-called complete set provided in [11] and performing some complicated calculations, it can be directly verified that the completeness relationship (86) and the orthogonal relation (75) in [11] are at all untenable. Hence, the complete set of squared Jost solutions have not been constructed successfully in [11].

4. Effects of perturbation on dark solitons

4.1. Expansion in the complete set of squared Jost solutions

Next, we shall solve the linearized inhomogeneous equation (36). In the TFR, equation (36) can be rewritten as

$$\left\{\mathrm{i}\partial_{t_0} - \mathbf{L}(z)\right\}\mathbf{q} = \mathbf{R}. \tag{71}$$

Expanding $\mathbf{q}$ in the complete set $\{\Phi(z,\zeta), \Psi(z,\zeta), \Phi_0(z), \Phi_1(z)\}$ as

$$|\mathbf{q}\rangle = \int_C \mathrm{d}\zeta\, q(\zeta)|\Phi(z,\zeta)\rangle + \int_C \mathrm{d}\zeta\, \widetilde{q}(\zeta)|\Psi(z,\zeta)\rangle + q_0|\Phi_0(z)\rangle + \widetilde{q}_0|\Phi_1(z)\rangle \tag{72}$$

and using the orthogonal relations (64)–(70), the ordinary differential equations and the corresponding initial conditions satisfied by expanding coefficients $q(\zeta)$, $\widetilde{q}(\zeta)$, q_0 and $\widetilde{q}_0$ can be obtained:

$$\widetilde{q}_{0t_0} = \langle\Psi_0(z)|\mathbf{R}\rangle, \qquad \widetilde{q}_0|_{t_0=0} = 0, \tag{73}$$

$$\mathrm{i}q_{0t_0} - 8k_1^3\widetilde{q}_0 = \langle\Psi_1(z)|\mathbf{R}\rangle, \qquad q_0|_{t_0=0} = 0, \tag{74}$$

$$\mathrm{i}q_{t_0}(\zeta) + 4\kappa(\lambda-\lambda_1)q(\zeta) = -\frac{\langle\Phi(z,\zeta)|\mathbf{R}\rangle}{2\pi a(\zeta)^2(1-\rho^4\zeta^{-4})}, \qquad q(\zeta)|_{t_0=0} = 0, \tag{75}$$

$$\mathrm{i}\widetilde{q}_{t_0}(\zeta) - 4\kappa(\lambda-\lambda_1)\widetilde{q}(\zeta) = \frac{\langle\Psi(z,\zeta)|\mathbf{R}\rangle}{2\pi a(\zeta)^2(1-\rho^4\zeta^{-4})}, \qquad \widetilde{q}(\zeta)|_{t_0=0} = 0. \tag{76}$$

It is noted that in the TER, the effective source $R = r[u] - \mathrm{i}u'$ is independent of t_0 since in the TFR, the zeroth-order approximate (adiabatic) solution u of the perturbed NLS$^+$ equation is independent of the rapid-variant time t_0 and may depend on the slow-variant time t_1 $(=\epsilon t)$ only through the time dependence of soliton parameters. Thus, the effective source vector $\mathbf{R}$ is also independent of t_0 in the TFR, and from equation (73) we have

$$\widetilde{q}_0 = -\mathrm{i}\langle\Psi_0(z)|\mathbf{R}\rangle t_0. \tag{77}$$

Obviously, this is a so-called secular term because $\widetilde{q}_0$ grows infinitely in time t_0. The secular condition to diminish this term is

$$\langle\Psi_0(z)|\mathbf{R}\rangle = \int_{-\infty}^{\infty} \mathrm{d}z\, \Psi_0(z)^A\mathbf{R} = 0. \tag{78}$$

Combining equations (77)–(78) with equation (74), we have

$$q_0 = -\mathrm{i}\langle\Psi_1(z)|\mathbf{R}\rangle t_0, \tag{79}$$

which is another secular term, and the corresponding secular condition is

$$\langle\Psi_1(z)|\mathbf{R}\rangle = \int_{-\infty}^{\infty} \mathrm{d}z\, \Psi_1(z)^A\mathbf{R} = 0. \tag{80}$$

4.2. Effects of perturbation on the soliton parameters

With the expression for the effective source, the secular conditions (78) and (80) can be rewritten as

$$\langle\Psi_0(z)|\mathrm{i}\mathbf{u}'\rangle = \langle\Psi_0(z)|\mathbf{r}\rangle \quad \text{or} \quad \int_{-\infty}^{\infty} \mathrm{d}z\, \Psi_0(z)^A\mathrm{i}\mathbf{u}' = \int_{-\infty}^{\infty} \mathrm{d}z\, \Psi_0(z)^A\mathbf{r}, \tag{81}$$

$$\langle\Psi_1(z)|\mathrm{i}\mathbf{u}'\rangle = \langle\Psi_1(z)|\mathbf{r}\rangle \quad \text{or} \quad \int_{-\infty}^{\infty} \mathrm{d}z\, \Psi_1(z)^A\mathrm{i}\mathbf{u}' = \int_{-\infty}^{\infty} \mathrm{d}z\, \Psi_1(z)^A\mathbf{r}, \tag{82}$$

in which

$$\mathbf{u}' = \begin{pmatrix} u' \\ \overline{u}' \end{pmatrix}, \qquad \mathbf{r} = \begin{pmatrix} r[u] \\ -\overline{r}[u] \end{pmatrix}, \tag{83}$$

$$u(z,t) = \mathrm{e}^{-\mathrm{i}\beta_1}\{\lambda_1 + \mathrm{i}k_1\tanh\theta_1\}, \qquad \theta_1 = k_1(z-z_c), \tag{84}$$

$u \equiv u(z,t)$ is the zeroth-order approximate (adiabatic) solution of perturbed NLS$^+$ equation, and z_c is a parameter describing the shift of the soliton centre with $z_c|_{t=0} = 0$. Since there exists the perturbation, the soliton parameters depend on the slow-variant time t_1 $(=\epsilon t)$. With the following expressions:

$$\theta_1' = \frac{k_1'}{k_1}\theta_1 - k_1 z_c', \tag{85}$$

$$e^{i\beta_1}\, iu = \lambda_1 + ik_1 \tanh\theta_1, \tag{86}$$

$$\mathrm{Im}[e^{i\beta_1}\, iu'] = \lambda_1' + k_1 \tanh\theta_1 \beta_1', \tag{87}$$

$$\mathrm{Re}[e^{i\beta_1}\, iu'] = \lambda_1\beta_1' - k_1' \tanh\theta_1 - k_1'\theta_1 \operatorname{sech}^2\theta_1 + k_1^2 z_c' \operatorname{sech}^2\theta_1, \tag{88}$$

we find that

$$-2ik_1\, e^{-i\beta_1}\langle\Psi_0(z)|i\mathbf{u}'\rangle = \int_{-\infty}^{\infty} d\theta_1 \operatorname{sech}^2\theta_1\, \mathrm{Im}(e^{i\beta_1}\, iu') = 2\lambda_1', \tag{89}$$

$$\frac{1}{2}k_1\, e^{-i\beta_1}\langle\Psi_1(z)|i\mathbf{u}'\rangle = \int_{-\infty}^{\infty} d\theta_1\, k_1 \operatorname{sech}\theta_1\, e^{\theta_1}\, \mathrm{Re}(e^{i\beta_1}\, iu') - \int_{-\infty}^{\infty} d\theta_1 \left(\lambda_1\theta_1 \operatorname{sech}^2\theta_1 + \frac{\lambda_1}{2}\operatorname{sech}^2\theta_1\, e^{2\theta_1}\right) \mathrm{Im}(e^{i\beta_1} iu') = -2\mathcal{L}\rho\rho' + \rho\rho' + \lambda_1 k_1\beta_1' + 2k_1^3 z_c', \tag{90}$$

where $2\mathcal{L} \longrightarrow \infty$ is the length of the system. This linear divergence comes from the fact that the dark solitons have infinite background energy. By calculating, we get the time-dependent relations of soliton parameters:

$$\epsilon\lambda_1' = \frac{1}{2}\int_{-\infty}^{\infty} d\theta_1 \operatorname{sech}^2\theta_1\, \mathrm{Im}(e^{i\beta_1}\epsilon r[u]), \tag{91}$$

$$-2\mathcal{L}\rho\epsilon\rho' + \rho\epsilon\rho' + \lambda_1 k_1\epsilon\beta_1' + 2k_1^3\epsilon z_c' = \int_{-\infty}^{\infty} d\theta_1\, k_1 \operatorname{sech}\theta_1\, e^{\theta_1}\, \mathrm{Re}(e^{i\beta_1}\epsilon r[u]) - \int_{-\infty}^{\infty} d\theta_1 \left(\lambda_1\theta_1 \operatorname{sech}^2\theta_1 + \frac{\lambda_1}{2}\operatorname{sech}^2\theta_1\, e^{2\theta_1}\right) \mathrm{Im}(e^{i\beta_1}\epsilon r[u]). \tag{92}$$

In fact, the secular condition (92) consists of two independent equations: the term diverging in the form of $2\mathcal{L}$ and the finite term in equation (92) must be separately equal to zero. Then, it is clear that the difficulties caused by the infinite background energy have been overcome. As only two out of λ_1, k_1, β_1 and ρ are independent ($\rho\, e^{i\beta_1} = \lambda_1 + ik_1$), we have obtained three independent equations to determine the evolutions of three independent soliton parameters (ρ, β_1 and z_c or λ_1, k_1 and z_c). For vanishing perturbation ($r \longrightarrow 0$ as $|x| \longrightarrow \infty$), there is no $2\mathcal{L}$ term on the right-hand side of equation (92), then $\rho' = 0$. For non-vanishing perturbation ($r \longrightarrow$ constant as $|x| \longrightarrow \infty$), the $2\mathcal{L}$ term will appear in the integrals on the right-hand side of equation (92), then $\rho' \neq 0$. It is noteworthy that equation (91) in this subsection is the same equation as equation (124) in [11] and that equation (92) in this subsection is different from equation (125) in [11], which can result in the difference between the evolution of some soliton parameters obtained in this paper and that obtained in [11].

4.3. The first-order correction

Considering the secular conditions (78) and (80), the expansion (72) becomes

$$|\mathbf{q}\rangle = \int_C d\zeta\, q(\zeta)|\Phi(z,\zeta)\rangle + \int_C d\zeta\, \widetilde{q}(\zeta)|\Psi(z,\zeta)\rangle. \tag{93}$$

The effective source $R = r[u] - \mathrm{i}u'$ can be determined according to the evolution relations of soliton parameters with the slow-variant time ϵt, and the expanding coefficients

$$\epsilon q(\zeta) = \epsilon q(\zeta, t_0) = -\frac{\langle \Phi(z,\zeta)|\epsilon \mathbf{R}\rangle}{8\pi\kappa(\lambda-\lambda_1)a(\zeta)^2(1-\rho^4\zeta^{-4})}(1-\exp(\mathrm{i}4\kappa(\lambda-\lambda_1)t_0)), \tag{94}$$

$$\epsilon \widetilde{q}(\zeta) = \epsilon \widetilde{q}(\zeta, t_0) = -\frac{\langle \Psi(z,\zeta)|\epsilon \mathbf{R}\rangle}{8\pi\kappa(\lambda-\lambda_1)a(\zeta)^2(1-\rho^4\zeta^{-4})}(1-\exp(-\mathrm{i}4\kappa(\lambda-\lambda_1)t_0)), \tag{95}$$

can be obtained from equations (75) and (76), respectively. Then, we get the formula calculating the first-order correction

$$\epsilon q(z,t) = \int_C \mathrm{d}\zeta\, \epsilon q(\zeta,t)\phi_1^2(z,\zeta) + \int_C \mathrm{d}\zeta\, \epsilon\widetilde{q}(\zeta,t)\psi_1^2(z,\zeta), \tag{96}$$

where the explicit expressions of $\phi_1(z,\zeta)$ and $\psi_1(z,\zeta)$ are given in appendix A. Here, we have replaced t_0 with t. Usually, the integrand of equation (96) is very complicated and is difficult to calculate exactly. Formula (96) is different from equation (129) in [11], the former contains two integral terms and the latter contains only one integral term.

5. Example: dark soliton evolution under damping

As an important example, we consider the damping NLS$^+$ equation with $\epsilon r[v] = -\mathrm{i}\gamma v$,

$$\mathrm{i}v_t - v_{xx} + 2(|v|^2 - \rho^2)v = -\mathrm{i}\gamma v, \tag{97}$$

where γ is the loss rate and $\gamma \sim \epsilon$. By employing the formulae provided above, one can find

$$\epsilon r[u] = -\mathrm{i}\gamma u, \qquad \mathrm{Re}(\mathrm{e}^{\mathrm{i}\beta_1}\epsilon r[u]) = \gamma k_1 \tanh\theta_1, \qquad \mathrm{Im}(\mathrm{e}^{\mathrm{i}\beta_1}\epsilon r[u]) = -\gamma\lambda_1, \tag{98}$$

$$\epsilon\lambda_1' = -\gamma\lambda_1, \qquad -2\mathcal{L}\rho\epsilon\rho' + \rho\epsilon\rho' + \lambda_1 k_1\epsilon\beta_1' + 2k_1^3\epsilon z_c' = 2\mathcal{L}\gamma\rho^2 - 2\gamma k_1^2 - \gamma\lambda_1^2. \tag{99}$$

In second equation in (99), the divergent and finite terms should be separately equal to zero, which results in the following three equations of soliton parameters:

$$\epsilon\lambda_1' = -\gamma\lambda_1, \qquad \epsilon\rho' = -\gamma\rho, \qquad \lambda_1 k_1\epsilon\beta_1' + 2k_1^3\epsilon z_c' = \gamma\rho^2 - 2\gamma k_1^2 - \gamma\lambda_1^2.$$

The solutions of the above equations are

$$\lambda_1(t) = \lambda_1(0)\,\mathrm{e}^{-\gamma t}, \qquad k_1(t) = k_1(0)\,\mathrm{e}^{-\gamma t}, \tag{100}$$

$$\rho(t) = \rho(0)\,\mathrm{e}^{-\gamma t}, \qquad \beta_1(t) = \beta_1(0), \tag{101}$$

$$z_c(t) = -\frac{\mathrm{e}^{\gamma t}-1}{2k_1(0)} \approx -\frac{\gamma t}{2k_1(0)}, \tag{102}$$

where $\beta_1(0)$, $\rho(0)$, $\lambda_1(0)$ and $k_1(0) = \sqrt{\rho(0)^2 - \lambda_1(0)^2}$ are the corresponding initial values. It can be verified that the other methods used in [11, 12, 15] can also yield the same results as above, expect for z_c. No shift of the soliton centre (z_c in this paper) can be predicted in [12, 15], while the shift of soliton centre predicted by us differs from that predicted in [11] in having a negative symbol.

Next, we calculate the first-order correction. With the above evolution equations of soliton parameters, the effective source is determined to be

$$\epsilon R = \epsilon r[u] - \mathrm{i}\epsilon u' = -\gamma\,\mathrm{e}^{-\mathrm{i}\beta_1}k_1\left(\theta_1 - \tfrac{1}{2}\right)\mathrm{sech}^2\,\theta_1, \tag{103}$$

and one has

$$\begin{aligned}\langle\Phi(z,\zeta)|\epsilon\mathbf{R}\rangle &= \int_{-\infty}^{\infty}\mathrm{d}z\,\Phi(z,\zeta)^A\epsilon\mathbf{R}\\ &= \frac{\gamma\,\mathrm{e}^{-\mathrm{i}\beta_1}(\zeta-\zeta_1)}{\zeta^2(\zeta-\overline{\zeta}_1)}\int_{-\infty}^{\infty}\mathrm{d}\theta_1\exp\left(\mathrm{i}2\frac{\kappa}{k_1}\theta_1\right)\left(\theta_1-\frac{1}{2}\right)\mathrm{sech}^2\theta_1\\ &\quad\times[(\zeta-\overline{\zeta}_1)(\zeta+\zeta_1)-2\mathrm{i}k_1\zeta\,\mathrm{sech}\,\theta_1\,\mathrm{e}^{-\theta_1}]\\ &= \frac{2\mathrm{i}\gamma\mathrm{e}^{-\mathrm{i}\beta_1}(\zeta-\zeta_1)\kappa\pi}{\zeta(\zeta-\overline{\zeta}_1)\sinh\left(\pi\frac{\kappa}{k_1}\right)} = \frac{2\mathrm{i}\gamma a(\zeta)\kappa\pi}{\zeta\sinh\left(\pi\frac{\kappa}{k_1}\right)},\end{aligned} \tag{104}$$

$$\begin{aligned}\langle\Psi(z,\zeta)|\epsilon\mathbf{R}\rangle &= \int_{-\infty}^{\infty}\mathrm{d}z\,\Psi(z,\zeta)^A\epsilon\mathbf{R}\\ &= \frac{\gamma\,\mathrm{e}^{-\mathrm{i}\beta_1}(\zeta-\zeta_1)}{\zeta^2(\zeta-\overline{\zeta}_1)}\int_{-\infty}^{\infty}\mathrm{d}\theta_1\exp\left(-\mathrm{i}2\frac{\kappa}{k_1}\theta_1\right)\left(\theta_1-\frac{1}{2}\right)\mathrm{sech}^2\theta_1\\ &\quad\times[(\zeta+\overline{\zeta}_1)(\zeta-\zeta_1)+2\mathrm{i}k_1\zeta\,\mathrm{sech}\,\theta_1\,\mathrm{e}^{-\theta_1}]\\ &= -\frac{2\mathrm{i}\gamma\,\mathrm{e}^{-\mathrm{i}\beta_1}(\zeta-\zeta_1)\kappa\pi}{\zeta(\zeta-\overline{\zeta}_1)\sinh\left(\pi\frac{\kappa}{k_1}\right)} = -\frac{2\mathrm{i}\gamma a(\zeta)\kappa\pi}{\zeta\sinh\left(\pi\frac{\kappa}{k_1}\right)},\end{aligned} \tag{105}$$

$$\epsilon q(\zeta,t) = -\frac{\mathrm{i}\gamma a(\zeta)^{-1}[1-\mathrm{e}^{\mathrm{i}4\kappa(\lambda-\lambda_1)t}]}{4\zeta(1-\rho^4\zeta^{-4})\sinh\left(\pi\frac{\kappa}{k_1}\right)},\qquad \epsilon\widetilde{q}(\zeta,t) = \frac{\mathrm{i}\gamma a(\zeta)^{-1}[1-\mathrm{e}^{-\mathrm{i}4\kappa(\lambda-\lambda_1)t}]}{4\zeta(1-\rho^4\zeta^{-4})\sinh\left(\pi\frac{\kappa}{k_1}\right)}. \tag{106}$$

Substituting equation (106) into equation (96), one finally obtains the first-order correction

$$\begin{aligned}\epsilon q(z,t) &= \int_C\frac{\mathrm{i}\gamma a(\zeta)^{-1}\,\mathrm{d}\zeta}{4\zeta(1-\rho^4\zeta^{-4})\sinh\left(\pi\frac{\kappa}{k_1}\right)}\big\{[1-\exp(-\mathrm{i}4\kappa(\lambda-\lambda_1)t)]\phi_1^2(z,\zeta)\\ &\quad-[1-\exp(\mathrm{i}4\kappa(\lambda-\lambda_1)t)]\psi_1^2(z,\zeta)\big\}.\end{aligned} \tag{107}$$

The result differs greatly from that in [11].

6. Conclusion

So far, we have successfully developed a direct perturbation theory for dark solitons through the construction and the rigorous proof of the complete sets $\{\Phi(z,\zeta), \Psi(z,\zeta), \Phi_0(z), \Phi_1(z)\}$ and $\{\Phi(z,\zeta)^A, \Psi(z,\zeta)^A, \Psi_0(z)^A, \Psi_1(z)^A\}$. Using secular conditions, we have also obtained the evolution equations of single-soliton parameters and overcome the difficulties caused by infinite background energy. Hence, the general procedure for the adiabatic (zeroth-order approximate) solution of the perturbed NLS^+ equation has been given by providing the evolution of soliton parameters, and a formula for calculating first-order corrections has been obtained.

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Appendix A. Explicit expressions of $\{\Phi(z,\zeta), \Psi(z,\zeta), \Phi_0(z), \Phi_1(z)\}$ and $\{\Phi(z,\zeta)^A, \Psi(z,\zeta)^A, \Psi_0(z)^A, \Psi_1(z)^A\}$

In the TFR, the explicit expressions of some Jost solutions are

$$\phi_1(z,\zeta) = (\zeta - \zeta_1 + \mathrm{i}k_1 \operatorname{sech}\theta_1\, \mathrm{e}^{-\theta_1}) \frac{\mathrm{e}^{-\mathrm{i}\beta_1}\, \mathrm{e}^{-\mathrm{i}\kappa z}}{\zeta - \overline{\zeta}_1}, \tag{A.1}$$

$$\phi_2(z,\zeta) = \{\mathrm{i}\rho\zeta^{-1}(\zeta - \zeta_1) - \mathrm{e}^{\mathrm{i}\beta_1}\, k_1 \operatorname{sech}\theta_1\, \mathrm{e}^{-\theta_1}\} \frac{\mathrm{e}^{-\mathrm{i}\beta_1}\, \mathrm{e}^{-\mathrm{i}\kappa z}}{\zeta - \overline{\zeta}_1}, \tag{A.2}$$

$$\psi_1(z,\zeta) = -\{\mathrm{i}\rho\zeta^{-1}(\zeta - \overline{\zeta}_1) + \mathrm{e}^{-\mathrm{i}\beta_1}\, k_1 \operatorname{sech}\theta_1\, \mathrm{e}^{-\theta_1}\} \frac{\mathrm{e}^{\mathrm{i}\kappa z}}{\zeta - \overline{\zeta}_1}, \tag{A.3}$$

$$\psi_2(z,\zeta) = (\zeta - \overline{\zeta}_1 - \mathrm{i}k_1 \operatorname{sech}\theta_1\, \mathrm{e}^{-\theta_1}) \frac{\mathrm{e}^{\mathrm{i}\kappa z}}{\zeta - \overline{\zeta}_1}. \tag{A.4}$$

From equations (26)–(27), (44), (46) and (59), the explicit expressions of $\{\Phi(z,\zeta), \Psi(z,\zeta), \Phi_0(z), \Phi_1(z)\}$ can be given by

$$\Phi(z,\zeta) = \begin{pmatrix} \{\zeta - \zeta_1 + \mathrm{i}k_1 \operatorname{sech}\theta_1\, \mathrm{e}^{-\theta_1}\}^2 \\ \{\mathrm{i}\rho\zeta^{-1}(\zeta - \zeta_1) - \mathrm{e}^{\mathrm{i}\beta_1} k_1 \operatorname{sech}\theta_1\, \mathrm{e}^{-\theta_1}\}^2 \end{pmatrix} \frac{\mathrm{e}^{-\mathrm{i}2\beta_1}\, \mathrm{e}^{-\mathrm{i}2\kappa z}}{(\zeta - \overline{\zeta}_1)^2}, \tag{A.5}$$

$$\Psi(z,\zeta) = \begin{pmatrix} \{\mathrm{i}\rho\zeta^{-1}(\zeta - \overline{\zeta}_1) + \mathrm{e}^{-\mathrm{i}\beta_1} k_1 \operatorname{sech}\theta_1\, \mathrm{e}^{-\theta_1}\}^2 \\ \{\zeta - \overline{\zeta}_1 - \mathrm{i}k_1 \operatorname{sech}\theta_1\, \mathrm{e}^{-\theta_1}\}^2 \end{pmatrix} \frac{\mathrm{e}^{\mathrm{i}2\kappa z}}{(\zeta - \overline{\zeta}_1)^2}, \tag{A.6}$$

$$\Phi_0(z) = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}2\beta_1} \\ -1 \end{pmatrix} \frac{1}{4} \operatorname{sech}^2\theta_1, \tag{A.7}$$

$$\Phi_1(z) = \begin{pmatrix} \{-\mathrm{i}\lambda_1\, \theta_1 \operatorname{sech}^2\theta_1 - \frac{\mathrm{i}}{2}\lambda_1 \operatorname{sech}^2\theta_1 + \mathrm{i}\overline{\zeta}_1 \operatorname{sech}\theta_1\, \mathrm{e}^{-\theta_1}\}\, \mathrm{e}^{-\mathrm{i}2\beta_1} \\ \mathrm{i}\lambda_1\, \theta_1 \operatorname{sech}^2\theta_1 + \frac{\mathrm{i}}{2}\lambda_1 \operatorname{sech}^2\theta_1 - \mathrm{i}\zeta_1 \operatorname{sech}\theta_1\, \mathrm{e}^{-\theta_1} \end{pmatrix}. \tag{A.8}$$

From equation (68) and equations (54)–(55) defining the adjoint states of $\Phi(z,\zeta)$ and $\Psi(z,\zeta)$, $\{\Phi(z,\zeta)^A, \Psi(z,\zeta)^A, \Psi_0(z)^A, \Psi_1(z)^A\}$ can be expressed explicitly as

$$\Phi(z,\zeta)^A = \begin{pmatrix} \{\zeta - \overline{\zeta}_1 - \mathrm{i}k_1 \operatorname{sech}\theta_1\, \mathrm{e}^{-\theta_1}\}^2 \\ \{\rho\zeta^{-1}(\zeta - \overline{\zeta}_1) - \mathrm{i}k_1\, \mathrm{e}^{-\mathrm{i}\beta_1} \operatorname{sech}\theta_1\, \mathrm{e}^{-\theta_1}\}^2 \end{pmatrix}^T \frac{-\mathrm{e}^{\mathrm{i}2\kappa z}}{(\zeta - \overline{\zeta}_1)^2}, \tag{A.9}$$

$$\Psi(z,\zeta)^A = \begin{pmatrix} \{\rho\zeta^{-1}(\zeta - \zeta_1) + \mathrm{i}k_1\, \mathrm{e}^{\mathrm{i}\beta_1} \operatorname{sech}\theta_1\, \mathrm{e}^{-\theta_1}\}^2 \\ \{\zeta - \zeta_1 + \mathrm{i}k_1 \operatorname{sech}\theta_1\, \mathrm{e}^{-\theta_1}\}^2 \end{pmatrix}^T \frac{\mathrm{e}^{-\mathrm{i}2\beta_1}\, \mathrm{e}^{-\mathrm{i}2\kappa z}}{(\zeta - \overline{\zeta}_1)^2}, \tag{A.10}$$

$$\Psi_0(z)^A = \begin{pmatrix} \mathrm{e}^{\mathrm{i}2\beta_1} \\ 1 \end{pmatrix}^T \frac{1}{4} \operatorname{sech}^2\theta_1, \tag{A.11}$$

$$\Psi_1(z)^A = \begin{pmatrix} \{\mathrm{i}\lambda_1\, \theta_1 \operatorname{sech}^2\theta_1 - \frac{\mathrm{i}}{2}\lambda_1 \operatorname{sech}^2\theta_1 + \mathrm{i}\overline{\zeta}_1 \operatorname{sech}\theta_1\, \mathrm{e}^{\theta_1}\}\, \mathrm{e}^{\mathrm{i}2\beta_1} \\ \mathrm{i}\lambda_1\theta_1 \operatorname{sech}^2\theta_1 - \frac{\mathrm{i}}{2}\lambda_1 \operatorname{sech}^2\theta_1 + \mathrm{i}\zeta_1 \operatorname{sech}\theta_1\, \mathrm{e}^{\theta_1} \end{pmatrix}^T. \tag{A.12}$$

Appendix B. The completeness of sets $\{\Phi(z,\zeta),\Psi(z,\zeta),\Phi_0(z),\Phi_1(z)\}$ and $\{\Phi(z,\zeta)^A,\Psi(z,\zeta)^A,\Psi_0(z)^A,\Psi_1(z)^A\}$

We start from the integral in equation (63), let

$$\frac{1}{2\pi}\int_C \frac{d\zeta}{a(\zeta)^2(1-\rho^4\zeta^{-4})}\{\Psi(z,\zeta)\Psi(z',\zeta)^A-\Phi(z,\zeta)\Phi(z',\zeta)^A\}=\begin{pmatrix}J_{11} & J_{12}\\ J_{21} & J_{22}\end{pmatrix}. \tag{B.1}$$

Using the explicit expressions (A.5)–(A.6) and (A.9)–(A.10) of squared Jost solutions and making a proper substitution ($\zeta \longrightarrow -\zeta$) in arranging the above integral, we have

$$J_{11}=\delta(z-z')+\int_C d\zeta\ f_{11}(\zeta),\qquad J_{12}=\int_C d\zeta\ f_{12}(\zeta), \tag{B.2}$$

$$J_{22}=\delta(z-z')+\int_C d\zeta\ f_{22}(\zeta),\qquad J_{21}=\int_C d\zeta\ f_{21}(\zeta), \tag{B.3}$$

where

$$\begin{aligned} f_{11}(\zeta)=\frac{\exp(i2\kappa(z-z'))}{2\pi(1-\rho^4\zeta^{-4})}\Bigg[&\left(1-\frac{ik_1}{\zeta+\zeta_1}\operatorname{sech}\theta_1\,e^{-\theta_1}\right)^2\left(1+\frac{ik_1}{\zeta-\bar{\zeta}_1}\operatorname{sech}\theta_1'\,e^{-\theta_1'}\right)^2\\ &-\left(i\zeta_1\zeta^{-1}+\frac{k_1}{\zeta-\bar{\zeta}_1}\operatorname{sech}\theta_1\,e^{-\theta_1}\right)^2\left(i\bar{\zeta}_1\zeta^{-1}-\frac{k_1}{\zeta-\zeta_1}\operatorname{sech}\theta_1'\,e^{-\theta_1'}\right)^2\\ &-(1-\rho^4\zeta^{-4})\Bigg], \end{aligned} \tag{B.4}$$

$$\begin{aligned} f_{22}(\zeta)=\frac{\exp(i2\kappa(z-z'))}{2\pi(1-\rho^4\zeta^{-4})}\Bigg[&\left(1-\frac{ik_1}{\zeta-\bar{\zeta}_1}\operatorname{sech}\theta_1\,e^{-\theta_1}\right)^2\left(1+\frac{ik_1}{\zeta-\zeta_1}\operatorname{sech}\theta_1'\,e^{-\theta_1'}\right)^2\\ &-\left(i\bar{\zeta}_1\zeta^{-1}-\frac{k_1}{\zeta+\zeta_1}\operatorname{sech}\theta_1\,e^{-\theta_1}\right)^2\left(i\zeta_1\zeta^{-1}+\frac{k_1}{\zeta+\bar{\zeta}_1}\operatorname{sech}\theta_1'\,e^{-\theta_1'}\right)^2\\ &-(1-\rho^4\zeta^{-4})\Bigg], \end{aligned} \tag{B.5}$$

$$\begin{aligned} f_{12}(\zeta)=\frac{\exp(i2\kappa(z-z'))}{2\pi(1-\rho^4\zeta^{-4})}\Bigg[&\left(i\rho\zeta^{-1}+\frac{e^{-i\beta_1}k_1}{\zeta-\bar{\zeta}_1}\operatorname{sech}\theta_1\,e^{-\theta_1}\right)^2\left(1+\frac{ik_1}{\zeta-\zeta_1}\operatorname{sech}\theta_1'\,e^{-\theta_1'}\right)^2\\ &-\left(1-\frac{ik_1}{\zeta+\zeta_1}\operatorname{sech}\theta_1\,e^{-\theta_1}\right)^2\left(i\rho\zeta^{-1}+\frac{e^{-i\beta_1}k_1}{\zeta+\bar{\zeta}_1}\operatorname{sech}\theta_1'\,e^{-\theta_1'}\right)^2\Bigg], \end{aligned} \tag{B.6}$$

$$\begin{aligned} f_{21}(\zeta)=\frac{\exp(-i2\beta_1)\,e^{i2\kappa(z-z')}}{2\pi(1-\rho^4\zeta^{-4})\zeta^2}\Bigg[&\left(i\bar{\zeta}_1-\frac{k_1\zeta}{\zeta+\zeta_1}\operatorname{sech}\theta_1\,e^{-\theta_1}\right)^2\left(1+\frac{ik_1}{\zeta+\bar{\zeta}_1}\operatorname{sech}\theta_1'\,e^{-\theta_1'}\right)^2\\ &-\left(1-\frac{ik_1}{\zeta-\bar{\zeta}_1}\operatorname{sech}\theta_1\,e^{-\theta_1}\right)^2\left(i\bar{\zeta}_1-\frac{k_1\zeta}{\zeta-\zeta_1}\operatorname{sech}\theta_1'\,e^{-\theta_1'}\right)^2\Bigg]. \end{aligned} \tag{B.7}$$

It is easy to see that $f(\zeta)$ ($=f_{11}(\zeta)$, $f_{22}(\zeta)$, $f_{12}(\zeta)$ or $f_{21}(\zeta)$) as a function of complex variable ζ is analytical everywhere except for the simple poles $\pm\rho$, $\pm i\rho$ and the double poles $\pm\zeta_1$, $\mp\bar{\zeta}_1$

and is bounded. Since $|f(\zeta)| \longrightarrow 0$ as $|\zeta| \longrightarrow \infty$, according to Jordan's lemma, it follows that

$$\int_C \mathrm{d}\zeta\ f(\zeta) = \begin{cases} 2\pi\mathrm{i}\{\mathrm{Res}\, f(\mathrm{i}\rho) + \mathrm{Res}\, f(\zeta_1) + \mathrm{Res}\, f(-\overline{\zeta}_1)\}, & \text{for} \quad z - z' > 0, \\ -2\pi\mathrm{i}\{\mathrm{Res}\, f(\rho) + \mathrm{Res}\, f(-\rho) + \mathrm{Res}\, f(-\mathrm{i}\rho) & \\ \quad + \mathrm{Res}\, f(-\zeta_1) + \mathrm{Res}\, f(\overline{\zeta}_1)\}, & \text{for} \quad z - z' < 0, \end{cases} \tag{B.8}$$

where $\mathrm{Res}\, f(\zeta_0)$ denotes the residue of $f(\zeta)$ at the pole ζ_0. According to residue theorem, we first calculate the residues of $f_{11}(\zeta)$, $f_{22}(\zeta)$, $f_{12}(\zeta)$ and $f_{21}(\zeta)$ at all poles, and then substituting them into the above formula (B.8), we get the following integrals:

$$\int_C \mathrm{d}\zeta\ f_{11}(\zeta) = \frac{\mathrm{i}}{4}[\lambda_1(\theta_1 - \theta_1')\,\mathrm{sech}^2\,\theta_1\,\mathrm{sech}^2\,\theta_1' + \lambda_1\,\mathrm{sech}^2\,\theta_1\,\mathrm{sech}^2\,\theta_1'$$
$$- \overline{\zeta}_1\,\mathrm{sech}^2\,\theta_1\,\mathrm{sech}\,\theta_1'\,\mathrm{e}^{\theta_1'} - \overline{\zeta}_1\,\mathrm{sech}\,\theta_1\,\mathrm{e}^{-\theta_1}\,\mathrm{sech}^2\theta_1'], \tag{B.9}$$

$$\int_C \mathrm{d}\zeta\ f_{22}(\zeta) = \frac{\mathrm{i}}{4}[-\lambda_1\,(\theta_1 - \theta_1')\,\mathrm{sech}^2\,\theta_1\,\mathrm{sech}^2\,\theta_1' - \lambda_1\,\mathrm{sech}^2\,\theta_1\,\mathrm{sech}^2\,\theta_1'$$
$$+ \zeta_1\,\mathrm{sech}^2\,\theta_1\,\mathrm{sech}\,\theta_1'\,\mathrm{e}^{\theta_1'} + \zeta_1\,\mathrm{sech}\,\theta_1\,\mathrm{e}^{-\theta_1}\,\mathrm{sech}^2\,\theta_1'], \tag{B.10}$$

$$\int_C \mathrm{d}\zeta\ f_{12}(\zeta) = \frac{\mathrm{i}}{4}\,\mathrm{e}^{-\mathrm{i}2\beta_1}[\lambda_1(\theta_1 - \theta_1')\,\mathrm{sech}^2\,\theta_1\,\mathrm{sech}^2\,\theta_1' + \lambda_1\,\mathrm{sech}^2\,\theta_1\,\mathrm{sech}^2\,\theta_1'$$
$$- \zeta_1\,\mathrm{sech}^2\,\theta_1\,\mathrm{sech}\,\theta_1'\,\mathrm{e}^{\theta_1'} - \overline{\zeta}_1\,\mathrm{sech}\,\theta_1\,\mathrm{e}^{-\theta_1}\,\mathrm{sech}^2\,\theta_1'], \tag{B.11}$$

$$\int_C \mathrm{d}\zeta\ f_{21}(\zeta) = \frac{\mathrm{i}}{4}\,\mathrm{e}^{\mathrm{i}2\beta_1}[-\lambda_1(\theta_1 - \theta_1')\,\mathrm{sech}^2\,\theta_1\,\mathrm{sech}^2\,\theta_1' - \lambda_1\,\mathrm{sech}^2\,\theta_1\,\mathrm{sech}^2\,\theta_1'$$
$$+ \zeta_1\,\mathrm{sech}\,\theta_1\,\mathrm{e}^{-\theta_1}\,\mathrm{sech}^2\,\theta_1' + \overline{\zeta}_1\,\mathrm{sech}^2\,\theta_1\,\mathrm{sech}\,\theta_1'\,\mathrm{e}^{\theta_1'}]. \tag{B.12}$$

Combining equation (B.1) with equations (A.7)–(A.8), (A.11)–(A.12), (B.2)–(B.3) and (B.9)–(B.12), we finally obtain the completeness relationship

$$\frac{1}{2\pi}\int_C \frac{\mathrm{d}\zeta}{a(\zeta)^2(1-\rho^4\zeta^{-4})}\{\Psi(z,\zeta)\Psi(z',\zeta)^A - \Phi(z,\zeta)\Phi(z',\zeta)^A\}$$
$$+ \Phi_0(z)\Psi_1(z')^A + \Phi_1(z)\Psi_0(z')^A = I\delta(z-z'). \tag{B.13}$$

This means that $\{\Phi(z,\zeta), \Psi(z,\zeta), \Phi_0(z), \Phi_1(z)\}$ and $\{\Phi(z,\zeta)^A, \Psi(z,\zeta)^A, \Psi_0(z)^A, \Psi_1(z)^A\}$ really construct the complete sets.

Appendix C. The orthogonality of sets $\{\Phi(z,\zeta), \Psi(z,\zeta), \Phi_0(z), \Phi_1(z)\}$ and $\{\Phi(z,\zeta)^A, \Psi(z,\zeta)^A, \Psi_0(z)^A, \Psi_1(z)^A\}$

Before the proof of the orthogonality of the complete set, we give some useful integral formulae derived by using the residue method in [10]

$$I_0(k) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}k\theta_1}\,\mathrm{d}\,\theta_1 = 2\pi\delta(k), \tag{C.1}$$

$$I_1(k) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}k\theta_1}\,\mathrm{sech}\,\theta_1\,\mathrm{e}^{-\theta_1}\,\mathrm{d}\,\theta_1 = 2\pi\delta(k) - \frac{\mathrm{i}\pi}{\sinh\frac{k\pi}{2}}, \tag{C.2}$$

$$I_2(k) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}k\theta_1}\,\mathrm{sech}^2\,\theta_1\,\mathrm{e}^{-2\theta_1}\,\mathrm{d}\,\theta_1 = 4\pi\delta(k) - \frac{2\mathrm{i}\pi}{\sinh\frac{k\pi}{2}} - \frac{k\pi}{\sinh\frac{k\pi}{2}}, \tag{C.3}$$

$$I_3(k) = \int_{-\infty}^{\infty} e^{ik\theta_1} \operatorname{sech}^3\theta_1\, e^{-3\theta_1}\, d\theta_1 = 8\pi\delta(k) - \frac{4i\pi}{\sinh\frac{k\pi}{2}} - \frac{3k\pi}{\sinh\frac{k\pi}{2}} + \frac{ik^2\pi}{2\sinh\frac{k\pi}{2}}. \tag{C.4}$$

From the definition (58) of the inner product and the explicit expressions (A.5), (A.9), we have

$$\begin{aligned}\langle\Phi(z,\zeta')|\Phi(z,\zeta)\rangle &= \int_{-\infty}^{\infty}\Phi(z,\zeta')^A\Phi(z,\zeta)\,dz\\ &= \frac{e^{-i2\beta_1}/k_1}{(\zeta'-\overline{\zeta}_1)^2(\zeta-\overline{\zeta}_1)^2}\{(\rho^4\zeta^{-2}\zeta'^{-2}-1)(\zeta-\zeta_1)^2(\zeta'-\overline{\zeta}_1)^2 I_0(k)\\ &\quad + 2ik_1(\zeta-\zeta_1)(\zeta'-\overline{\zeta}_1)[2ik_1(\rho^2\zeta^{-1}\zeta'^{-1}-1)+(\zeta-\zeta')(1-\rho^4\zeta^{-2}\zeta'^{-2})]I_1(k)\\ &\quad + k_1^2[4(\zeta-\zeta_1)(\zeta'-\overline{\zeta}_1)(\rho^2\zeta^{-1}\zeta'^{-1}-1)+\zeta^2(1-\rho^4\zeta^{-4})+\zeta'^2(1-\rho^4\zeta'^{-4})\\ &\quad + 2\overline{\zeta}_1(\rho^2\zeta^{-1}-\zeta')+2\zeta_1(\rho^2\zeta'^{-1}-\zeta)]I_2(k)\\ &\quad + 2ik_1^3(\zeta-\zeta')(1-\rho^2\zeta^{-1}\zeta'^{-1})I_3(k)\},\end{aligned} \tag{C.5}$$

in which $k = [\zeta'-\zeta+\rho^2(\zeta^{-1}-\zeta'^{-1})]/k_1$. Inserting equations (C.1)–(C.4) into equation (C.5), we get

$$\begin{aligned}\langle\Phi(z,\zeta')|\Phi(z,\zeta)\rangle &= e^{-i2\beta_1}\frac{(\zeta-\zeta_1)^2}{(\zeta-\overline{\zeta}_1)^2}2\pi(\rho^4\zeta^{-4}-1)\delta(\zeta-\zeta')\\ &= -a(\zeta)^2(1-\rho^4\zeta^{-4})2\pi\delta(\zeta'-\zeta)\end{aligned} \tag{C.6}$$

through a series of calculations, that is, the orthogonal relation (64) has been verified. The orthogonal relations (65)–(68) can be derived in the same way, and the orthogonal relations (69)–(70) are easily obtained by straightforward calculations.

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